

Reconstruction of 2-convex polyominoes

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There are many notions of discrete convexity of polyominoes (namely hv -convex [1], Q -convex [2], L -convex polyominoes [5]) and each one has been deeply studied. One natural notion of convexity on the discrete plane leads to the definition of the class of hv -convex polyominoes, that is polyominoes with consecutive cells in rows and columns. In [1] and [6], it has been shown how to reconstruct in polynomial time hv -convex polyominoes from their horizontal and vertical projections. In addition to that, hv -convex polyominoes have been characterized by the presence, between each pair of its cells, of an internal path having at most two kinds of unit steps among the four possible ones, i.e. north, south, east and west steps (such a path is called monotone). For each $k \in \mathbb{N}$, we consider the hv -convex polyominoes where each couple of cells can be connected by a monotone path having at most k changes of direction, and we call them k -convex polyominoes. Since each hv -convex polyomino P has a finite number of pairs of cells, and so a finite number of monotone paths connecting them, then there exists an integer k such that P is k' -convex, for each $k' \geq k$. Thus, the families of k -convex polyominoes forms a hierarchy of hv -convex polyominoes. When the value of k is equal to 1 we have the so called L -convex polyominoes, where this terminology is motivated by the L -shape of the path having one single change of direction that connects any two of its cells. This notion of L -convex polyominoes has been considered by several points of view. In particular it has been shown that they are characterized by their horizontal and vertical projections [4]. Other tomographical and combinatorial aspects of L -convex polyominoes have been analyzed in [3] and [5].

Regarding 2-convex polyominoes (see Fig.1), they are geometrically more com-

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plex and there is no result concerning their direct reconstruction. On the other side, Duchi, Rinaldi, and Schaeffer enumerate this class using a purely analytical approach in [7], but their enumeration technique gives no idea for the tomographical reconstruction from two projections. So in the last years, some subclasses of 2-convex polyominoes have been considered, for instance, the class of centered polyominoes, i.e. those convex polyominoes having at least one row running from the left to the right side of its minimal bounding rectangle. A reconstruction algorithm for centered polyominoes is given in [6].

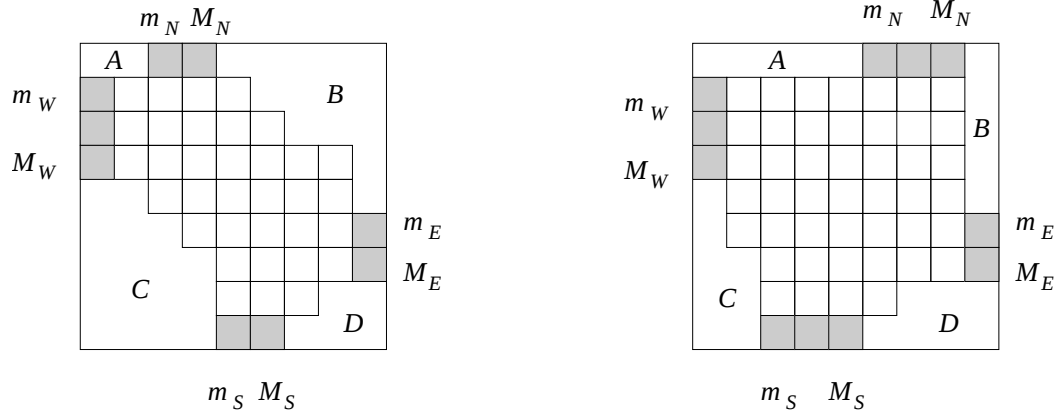


Fig. 1. An element of the class $\mathfrak{S}$ on the left and one of the class γ' on the right. The cells of the four feet are highlighted in both the polyominoes.

In this paper we furnish a polynomial time algorithm to solve the reconstruction problem for 2-convex polyominoes, which can be stated as follows:

Reconstruction (H, V)

Input: two integer vectors H and V .

Task: reconstruct a 2-convex polyomino whose horizontal and vertical projections are H and V , respectively, if it exists, otherwise give FAILURE.

In each convex polyomino P , we can define the N -foot to be the set of cells of P that lie in its first row. Note that, by convexity, the cells of the N -foot form a bar, and let us indicate by $(1, m_N)$ and $(1, M_N)$ its two extremal points, and sometimes, by abuse of notation, simply m_N and M_N . Analogously, we define the S -foot, W -foot, and E -foot of P , and their extremal points (see Fig. 1).

We define the following classes (see Fig. 1) that provide, together with the class of centered polyominoes, a partition of the 2-convex polyominoes: let $\mathcal{C}$ and $\mathcal{C}_2$ be the classes of convex and 2-convex polyominoes, respectively

- $\mathfrak{S} = \{P \in \mathcal{C}_2 \mid M_N < m_S \text{ and } M_W < m_E\};$
- $\mathfrak{S}' = \{P \in \mathcal{C}_2 \mid M_S < m_N \text{ and } M_E < m_W\};$
- $\gamma = \{P \in \mathcal{C} \mid M_N < m_S \text{ and } M_E < m_W\};$
- $\gamma' = \{P \in \mathcal{C} \mid M_S < m_N \text{ and } M_W < m_E\}.$

The classes γ and γ' can be reconstructed in polynomial time from their horizontal and vertical projections, by means of an algorithm defined in [8].

Furthermore, the classes $\mathfrak{S}$ and $\mathfrak{S}'$ coincide up to horizontal symmetry, so they are equivalent from a tomographical perspective, and to solve the reconstruction problem for 2-convex polyominoes is then sufficient to solve it for the class $\mathfrak{S}$.

A geometrical characterization of the class $\mathfrak{S}$

The reconstruction of the 2-convex polyominoes uses one of their geometrical characterizations that bases on the concept of maximal rectangle: given a convex polyomino P , we consider its south-east corners $d_1, \dots, d_r$ whose rows and columns indexes are greater than M_E and M_S , respectively (see Fig. 2, (a)). For each of these cells, say d_k , with $1 \leq k \leq r$, we consider the maximal rectangle entirely contained in P and having d_k as lower leftmost corner, and let i_k be its upper rightmost corner. Now, we consider the maximal rectangle having i_k as lower leftmost corner, and both the upper rightmost corner b_k and the lower leftmost corner c_k contained in P . Let a_k be the upper rightmost corner of this last rectangle (see Fig. 2, (b)). The following characterization holds: P is a 2-convex polyomino if and only if there are no cells of P having the row and column indexes less than those of a_k .

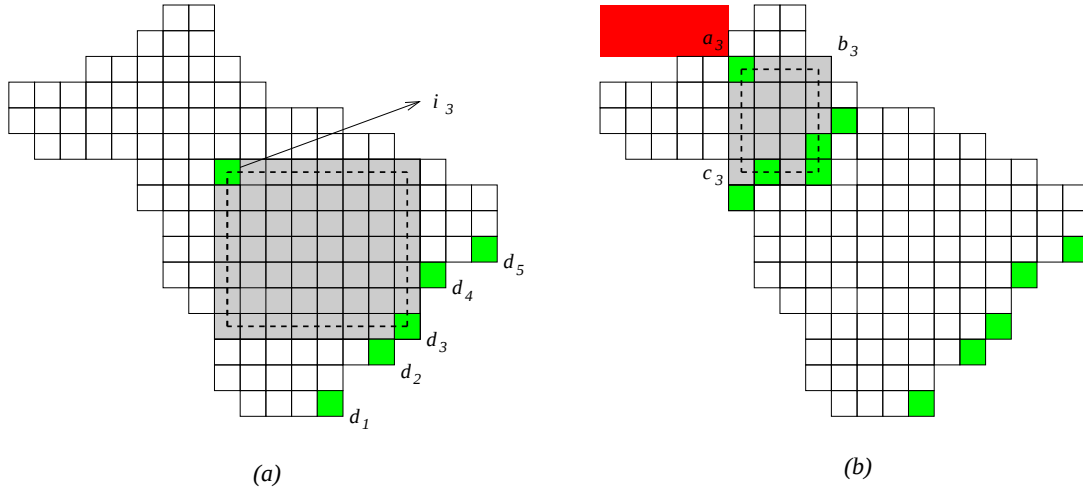


Fig. 2. The geometrical characterization of a 2-convex polyomino in $\mathfrak{S}$ using maximal rectangles. The rectangles and the cells concerning the corner d_3 are highlighted.

In [6], it is defined an algorithm to quickly reconstruct an hv -convex polyomino from its horizontal and vertical projections: the authors' idea relies on the possibility of using a 2-SAT formula $F_{k,l}(H, V)$, i.e. a boolean formula in conjunctive normal form with at most two literals in each clause, to express the geometrical characterization of an hv -convex polyomino in terms of four disjoint hv -convex zones whose union forms the exterior of the polyomino.

Following the strategy used in this work, we give a characterization of the polyominoes in $\mathfrak{S}$ adding to a slightly modified version of $F_{k,l}(H, V)$ that characterizes the hv -convexity, a polynomial number of new clauses to strengthen this constraints till expressing the 2-convexity in terms of maximal rectangles and paths as defined above; the obtained formula is addressed to as $\mathfrak{S}(H, V)$. Unfortunately, these new clauses prevent $\mathfrak{S}(H, V)$ from being a 2 – SAT formula any more, however it still consists only of Horn clauses. Since it can be computed a valuation for each set of Horn clauses in polynomial time by standard resolution methods, if it exists, then the following results hold

Theorem 1 *$\mathfrak{S}(H, V)$ is satisfiable if and only if there exists a polyomino P in $\mathfrak{S}$ having H and V as horizontal and vertical projections, respectively.*

Theorem 2 *If there exists a valuation for the formula $\mathfrak{S}(H, V)$, then it can be computed in polynomial time with respect to the dimensions m and n of the input instance.*

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